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Debt Ratio Optimization as an Alternative Approach to Capital Structure Determination Using the Lagrange Method

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Abstract
This study develops a mathematical optimization model for determining the optimal debt ratio in a firm's capital structure using the Lagrange method. By expressing equity value as a function of leverage, the model identifies a structural asymptote that divides the feasible region into positive and negative equity outcomes, thereby eliminating the possibility of an interior maximum and mathematically refuting the central prediction of the Trade-Off Theory. The first-order condition produces two critical points, but only one lies within the positive region and is shown through second-order analysis to be a local minimum rather than an optimum. Numerical simulation and empirical validation further demonstrate that optimal equity value arises under two extreme leverage conditions: zero leverage for conservative strategies and high leverage approaching the asymptote for aggressive strategies. These results establish a new analytical foundation for capital structure theory, provide a basis for developing more complex econometric models, and offer practical guidance for firms in prioritizing leverage decisions according to their risk preferences and strategic objectives.

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1. Introduction

The determination of an appropriate capital structure, particularly the proportion between debt and equity financing, remains one of the most extensively examined subjects in corporate finance. Capital structure is widely recognized as a fundamental determinant of firm performance, risk exposure, and market valuation. Seminal theoretical frameworks, including the Modigliani Miller theorem, the Static Trade Off Theory, and the Pecking Order Theory, emphasize that leverage influences the cost of capital through tax advantages, the probability of financial distress, and information asymmetry.

Recent empirical studies reinforce the notion that leverage exerts significant influence on firm value across different economic environments and institutional settings. For example, Tharavanij (2021) demonstrates that an optimal book value debt ratio exists that minimizes the weighted average cost of capital and enhances corporate valuation [1]. Ahmed et al. (2024) similarly provide evidence from seventy eight publicly listed firms that capital structure choices have material implications for profitability and overall corporate performance [2]. Bui et al. (2023) also find a positive relationship between debt and firm value, although without formulating a mathematical optimization model to determine the optimal level of leverage [3].

More recent studies further strengthen the empirical relevance of optimal leverage. Arhinful (2025) proposes a Fibonacci based optimal capital structure framework and identifies an optimal capital structure of approximately 61.8 percent debt and 38.2 percent equity, reaffirming the value maximizing role of leverage under certain conditions [4]. Likewise, Nguyen (2025) finds that firm performance follows a nonlinear relationship with leverage and reports that the global optimal threshold for firm value occurs near a debt ratio of 57.28 percent, while the optimal threshold for return on assets occurs near 24.40 percent [5]. These findings collectively highlight that leverage materially affects corporate valuation, although the precise mechanism and optimal point remain a subject of active debate.

Despite the breadth of empirical evidence, the methods traditionally used in capital structure research rely predominantly on regression based or econometric estimation approaches that identify statistical correlations rather than providing an explicit optimization mechanism. Existing tools such as target adjustment models, dynamic partial adjustment models, and tax bankruptcy trade off simulations attempt to approximate leverage targets but remain constrained by estimation bias, multicollinearity, omitted variable concerns, and sensitivity to modeling assumptions. More importantly, these approaches rarely integrate firm specific constraints such as interest coverage requirements, risk premium behavior, and distress cost escalation into a unified optimization framework. As a result, the literature continues to lack mathematically rigorous models capable of generating a closed form optimal debt ratio that remains theoretically and operationally consistent.

To address these limitations, the present study develops an explicit optimization model using the Lagrange multiplier method to formulate the determination of optimal leverage as a constrained maximization problem. This approach enables the integration of core financial components, including tax benefits, marginal distress costs, and interest coverage constraints, into a single analytical expression. Unlike conventional empirical approaches, the Lagrangian method provides a systematic and theoretically grounded mechanism for deriving a closed form solution for the optimal level of debt.

A key contribution of this research arises from the mathematical behavior of the constructed objective function. The function exhibits a convex curvature, such that the first order condition identifies a minimum rather than a maximum. This result implies that under the stated constraints, no interior maximum value of the firm exists, thereby challenging a core assumption of the Trade Off Theory which predicts a concave value function with an interior optimum. Although the model does not yield a maximum solution, the identification of the minimum critical point remains economically meaningful, as it delineates a region of minimized firm value that firms should strategically avoid. This insight produces two strategic financing implications: firms may prioritize equity financing to operate on the left side of the minimum point, or prioritize debt financing to position themselves on the right side of the minimum point.

The objectives of this study are therefore twofold. First, to construct and formalize a mathematical model for determining optimal leverage using the Lagrange multiplier method. Second, to evaluate empirically how the convex nature of the objective function affects real world leverage behavior relative to actual corporate data. By advancing an analytical optimization model and identifying boundary conditions under which classical theory fails, this study contributes to the refinement of capital structure theory and offers actionable insights for corporate decision makers seeking to optimize financing strategies.

2. Method

2.1. The Overall Idea of the Optimization Model

This chapter presents the conceptual foundation underlying the development of the optimization model used to determine the proportion of debt and equity financing. The primary objective of the model is to evaluate how different financing compositions influence equity value and to identify strategic regions of capital structure based on the mathematical characteristics of the objective function.

The model begins with a clearly defined optimization target, namely the maximization of equity value derived from the firm's financing composition. By incorporating the cost of debt, the cost of equity, the corporate tax rate, and growth assumptions, the model captures the fundamental financial mechanisms that drive changes in equity value. The interaction between these financial parameters and the financing weights forms the core basis for constructing the optimization framework.

To ensure theoretical consistency and mathematical validity, the framework introduces a fundamental structural constraint: the sum of the debt weight and equity weight must equal one. This constraint ensures that the model accurately reflects the proportional nature of corporate financing and restricts the solution space to feasible capital structure combinations.

In addition, this study deliberately adopts the Lagrange multiplier method due to its flexibility and openness toward the incorporation of additional constraints. Firms may face different financial limitations such as interest coverage requirements, internal risk thresholds, or regulatory restrictions. A Lagrangian-based structure allows these constraints to be integrated seamlessly without altering the fundamental shape of the objective function. This feature makes the model expandable and adaptable to various corporate environments, enabling its continued development as new constraints or conditions emerge.

The Lagrange multiplier method is therefore employed to unify the objective function and the constraint within a single analytical framework, enabling the derivation of the critical point for the optimization problem.

Based on this configuration, the model assesses how changes in financing weights affect equity value. Subsequent mathematical analysis reveals that the objective function is convex, meaning that the critical point produced by the first order condition corresponds to a minimum rather than a maximum. This property indicates that the model does not yield a unique optimal leverage point within the interior of the domain. Instead, it identifies a minimum equity value region that firms should avoid. As a result, the model provides two strategic financing directions that firms may pursue depending on their financial conditions and risk preferences.

The overall framework of research ideas can be summarized as follows:

1) Establishment of the optimization objective aimed at maximizing equity value
2) Identification of the key financial parameters influencing the objective function
3) Formulation of the proportional financing constraint
4) Adoption of the Lagrange method as an open and flexible mechanism for incorporating additional constraints
5) Integration of the objective function and constraint through the Lagrangian formulation
6) Derivation and evaluation of the critical point of the model
7) Interpretation of the convexity of the objective function and identification of strategic financing regions

This conceptual structure forms the basis for the mathematical construction presented in the next chapter. Through this systematic approach, the model provides a theoretically grounded and analytically clear method for assessing the impact of financing composition on a firm's equity value.

2.2. Construction of the Optimization Model

This chapter presents the construction of the optimization model used to evaluate how the composition of debt and equity financing affects the equity value of the firm. The development of the model follows a systematic and coherent approach that includes the establishment of modeling principles, the derivation of the objective function from firm valuation theory, the formulation of the constraint system, and the construction of an analytical framework that enables the rigorous application of the Lagrange multiplier method.

2.3. Fundamental Principles of Model Development

The optimization model is developed based on four fundamental principles designed to ensure financial relevance, mathematical validity, and structural flexibility.

2.3.1. Principle of Financial Representativeness

The model incorporates key financial parameters that directly influence equity value, including the cost of debt, cost of equity, corporate tax rate, cash flow growth rate, the weight of debt, the weight of equity, and the firm's capacity to generate free cash flow. These variables are grounded in modern corporate finance theory, which emphasizes that capital structure affects firm value through changes in aggregate cost of capital, risk exposure, and the sensitivity of cash flows to financing decisions. By integrating these fundamental variables, the model captures the essential dynamics that link financing composition to equity value both theoretically and empirically.

2.3.2. Principle of Elegant Structural Simplicity

To maintain analytical clarity, the model adopts a single core structural constraint, namely that the weights of debt and equity must sum to one. This proportional constraint reflects the total capital structure of the firm and provides mathematical stability in the optimization process. Additional constraints can be incorporated in future developments without altering the foundational structure of the model.

2.3.3. Principle of Mathematical Consistency and Validity

To be compatible with the Lagrange multiplier method, the objective function must be continuous, differentiable, and free from singularities within the feasible domain. Accordingly, the model requires the denominator of the valuation function to remain strictly positive. This ensures that all components of the objective function are properly defined and can be evaluated analytically.

2.3.4. Principle of Model Openness to Extensions

The Lagrange method is selected because of its capability to accommodate additional constraints such as risk limits, interest coverage requirements, or regulatory restrictions. This flexibility allows the model to be expanded and adapted to the specific financial conditions of different firms without altering the core structure of the objective function.

2.4. Theoretical Foundation of the Objective Function

The optimization model is grounded in Value Maximization Theory, which posits that the ultimate objective of corporate management is to maximize firm value rather than merely maximizing short term earnings. Within this framework, equity value represents the present value of expected future cash flows adjusted for the firm's financing structure and risk profile.

Damodaran (2012) [6, p. 198] provides a comprehensive valuation framework in which equity value can be expressed as:

Table 1. From Operating Asset Value to Equity Value Step

Step	Output
Discount the free cash flow to the firm at the cost of capital to get . . .	Value of operating assets of the firm +
Add the value of any assets whose earnings are not part of operating income	+ Cash and marketable securities + Value of minority holdings in other companies + Value of idle or unutilized assets
Subtract nonequity claims on the company	– Value of interest-bearing debt – Present value of operating lease commitments – Estimated value of minority interests in consolidated companies – Unfunded health care or pension obligations – Expected litigation payouts
To get to value of equity	= Value of equity

The expression can be formulated as follows:

$$Equity\ Value = Value\ of\ Operating\ Assets + Cash - Debt \quad (1)$$

Under a discounted cash flow approach with Gordon growth model, this expression becomes:

$$EV = \frac{FCF(1+g)}{WACC - g} + Cash - Debt \quad (2)$$

where WACC depends explicitly on capital structure:

$$WACC = k_d\omega_d(1-T) + k_e\omega_e + k_p\omega_p \quad (3)$$

Thus:

$$EV = \frac{FCF(1+g)}{k_d\omega_d(1-T) + k_e\omega_e + k_p\omega_p - g} + Cash - Debt \quad (4)$$

Where

$WACC = Weight\ Average\ Cost\ of\ Capital$

$FCF = Free\ Cash\ Flow$

$k_d = Cost\ of\ Debt$

$k_e = Cost\ of\ Equity$

$k_p = Cost\ of\ Preferen\ Stock$

$\omega_d = Weight\ of\ Debt$

$\omega_e = Weight\ of\ Equity$

$\omega_p = Weight\ of\ Preferen\ Stock$

$EV = Equity\ Value$

This formulation shows that equity value is directly affected by the weights of debt, equity, and preferred shares.

2.5. Transformation of Capital Structure in Equity Valuation

To eliminate Debt and Equity as independent variables and replace them with proportional weights, the debt component is expressed as:

$$Debt = \omega_d(Debt + Equity) \quad (5)$$

This transformation is consistent with the valuation approach adopted in this study, which is based on book value rather than market value. Under a book-value framework, the debt ratio is computed using interest-bearing debt divided by the sum of interest-bearing debt and book value of equity. This definition aligns the optimization model with the underlying accounting values recorded in financial statements and ensures consistency with the variables used in the valuation function. Book value is preferred because it provides stability, avoids short-term market volatility, and reflects the actual committed capital structure of the firm.

Substituting the book-value representation of debt into the valuation function yields:

$$V = \frac{F(1+g)}{k_d\omega_d(1-T) + k_e\omega_e + k_p\omega_p - g} + C - \omega_d(\mathrm{D}+\mathrm{E}) \quad (6)$$

where the substitutions:
$F = FCF$
$C = Cash \ \& \ Equivalent$
$D = Debt$
$E = Equity$
are used for analytical convenience.

The resulting equity-value function becomes:

$$EV = \frac{F(1+g)}{k_d\omega_d(1-T) + k_e\omega_e + k_p\omega_p - g} + C - \omega_d(\mathrm{D} + \mathrm{E}) \tag{7}$$

This transformation produces an objective function fully expressed in terms of financing weights, facilitating optimization and allowing the model to incorporate additional constraints without altering its structural foundation. Furthermore, the use of book-value ratios ensures that the optimization results are directly linked to the firm's internal financing structure, making the model both theoretically coherent and practically applicable.

2.6. Formulation of the Objective Function and Constraint System

1) **General Objective Function.** The general objective function is expressed as:

$$f(\omega_d, \omega_e, \omega_p) = \frac{F(1+g)}{k_d\omega_d(1-T) + k_e\omega_e + k_p\omega_p - g} + C - \omega_d(\mathrm{D} + \mathrm{E}) \tag{8}$$

2) **General Constraint.** The general capital-structure constraint is:

$$\omega_d + \omega_e + \omega_p = 1 \tag{9}$$

3) **Simplified Two-Component Structure.** Given the focus on the two primary financing sources (debt and equity), the model simplifies the constraint to:

$$\omega_d + \omega_e = 1 \tag{10}$$

The corresponding objective function becomes:

$$f(\omega_d, \omega_e) = \frac{F(1+g)}{k_d\omega_d(1-T) + k_e\omega_e - g} + C - \omega_d(\mathrm{D} + \mathrm{E}) \tag{11}$$

Equation (10) serves as the foundational objective function used in the Lagrangian optimization.

1) **Construction of the Lagrangian Model.** To unify the objective function and constraints within a single analytical framework, the Lagrangian function is formulated as:

$$\mathcal{L}(\omega_d, \omega_e, \lambda) = \frac{F(1+g)}{k_d\omega_d(1-T) + k_e\omega_e - g} + C - \omega_d(\mathrm{D} + \mathrm{E}) + \lambda(\omega_d + \omega_e - 1) \tag{12}$$

The Lagrange multiplier λ captures the interplay between the objective function and the proportional constraint, enabling the derivation of first-order conditions and curvature analysis. This formulation establishes the analytical foundation for the optimization process presented in the next chapter.

2.7. Optimization Method and Critical-Point Analysis

2.7.1. First-Order Optimality Condition and Critical-Point Formula

To optimize equation (11), stationary point is necessary to be determined using first order optimality condition.

$$f(\omega_d, \omega_e) = \frac{F(1+g)}{k_d\omega_d(1-T) + k_e\omega_e - g} + C - \omega_d(\mathrm{D} + \mathrm{E}) \tag{13}$$

To carry out a neater calculation in equation … (11), variable A is defined in equation (14).

$$A(\omega_d, \omega_e) = k_d\omega_d(1-T) + k_e\omega_e - g \tag{14}$$

$$thus \rightarrow \ f(\omega_d, \omega_e) = \frac{F(1+g)}{A(\omega_d, \omega_e)} + C - \omega_d(D + E)$$

$$f(\omega_d, \omega_e) = F(1+g)[A(\omega_d, \omega_e)]^{-1} + C - \omega_d(D + E)$$

As the optimisation is done only with respect to ω_d, partial differentiation is done to equation (11) with respect to ω_d. Partial differentiation is being done in form of equation (15) using variable A partial derivative form and the partial differentiation for variable A is done separately in equation (16).

$$\frac{\partial f(\omega_d,\omega_e)}{\partial \omega_d} = F(1+g)(-1)[A(\omega_d,\omega_e)]^{-2}\left(\frac{\partial A(\omega_d,\omega_e)}{\partial \omega_d}\right) - (D+E) \quad (15)$$

$$\frac{\partial A(\omega_d,\omega_e)}{\partial \omega_d} = k_d(1-T) + k_e\frac{\partial \omega_e}{\partial \omega_d} \quad (16)$$

Minding that $\omega_e$is still correlated with both equation (15) and (16), operations below using equation (10) as a boundary condition is required.

$$\omega_e = 1-\omega_d \; ; \; \frac{\partial \omega_e}{\partial \omega_d} = -1$$

$$A(\omega_d) = k_d\omega_d(1-T) + k_e(1-\omega_d) - g \; ; \; \frac{\partial A(\omega_d)}{\partial \omega_d} = k_d(1-T) + k_e(-1)$$

$$A(\omega_d) = \omega_d[k_d(1-T) - k_e] + k_e - g \; ; \; \frac{\partial A(\omega_d)}{\partial \omega_d} = k_d(1-T) - k_e \quad (17)$$

Substituting equation (15), and (16), the form of first partial derivative of $f(\omega_d)$ is written below as equation (18).

$$\frac{\partial f(\omega_d)}{\partial \omega_d} = F(1+g)(-1)[k_d\omega_d(1-T) + k_e\omega_e - g]^{-2}(k_d(1-T) - k_e) - (D+E) \quad (18)$$

With the Theorem of First Order optimality Condition, the stationary condition is met when (19) below is met. Thus, equating equations (18) and (19) giving equation (20).

$$\frac{\partial f(\omega_d)}{\partial \omega_d} = 0 \quad (19)$$

$$\frac{D+E}{F(1+g)[k_e - k_d(1-T)]} = [k_d\omega_d(1-T) + k_e\omega_e - g]^{-2} \quad (20)$$

Inexact solution for Equation (20) represented below in form of the equation (21) below. As there are two solutions for ω_d, these two solutions ($\omega_{d,1}$ dan $\omega_{d,2}$) is generalized as $\omega_{d(1,2)}$. As well as it is being done the same for $\omega_{e(1,2)}$.

$$k_d\omega_{d(1,2)}(1-T) + k_e\omega_{e(1,2)} - g = \pm\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} \quad (21)$$

Using the constraint from … (10), equation … (21) can be solved as below;

$$k_d\omega_{d(1,2)}(1-T) + k_e\left(1-\omega_{d(1,2)}\right) - g = \pm\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}}$$

$$\omega_{d(1,2)}(k_d(1-T) - k_e) + k_e - g = \pm\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}}$$

$$-\left(k_e - k_d(1-T)\right)\omega_d = \pm\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)$$

$$\omega_{d(1,2)} = \frac{\pm\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)}{-\left(k_e - k_d(1-T)\right)}$$

Separating the two solutions as below giving equation …(22) and …(23).

$$\omega_{d,1} = \frac{\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)}{-\left(k_e - k_d(1-T)\right)} \quad (22)$$

$$\omega_{d,2} = \frac{-\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)}{-\left(k_e - k_d(1-T)\right)} \quad (23)$$

Equation (22) and (23) is multiple solutions from the quadratic equation and only one point is global maxima point. The finding of this global maximum is done in the next sub-section.

2.7.2. Sign Behavior of the Objective Function Around the Asymptote

After substituting the constraint $\omega_e = 1 - \omega_d$, the objective function in equation (13) can be written in the compact form

$$f(\omega_d) = \frac{A}{a\omega_d + b} + C - \omega_d s \tag{24}$$

where
$A = F(1+g)$,
$a = k_d(1-T) - k_e < 0$,
$b = k_e - g$,
$s = D + E > 0$.

The vertical asymptote of the function occurs when the denominator of the fractional term equals zero, i.e.,

$$a\omega_d + b = 0 \Rightarrow \omega_d^{\text{asym}} = -\frac{b}{a} = \frac{g - k_e}{a} \tag{25}$$

Because $a < 0$, the sign of the denominator changes as follows:
As $\omega_d \to \omega_d^{\text{asym}-}$ (approaching the asymptote from the left), $a\omega_d + b \to 0^+$.
Hence

$$\frac{A}{a\omega_d + b} \to +\infty \tag{26}$$

and thus

$$f(\omega_d) \to +\infty \tag{27}$$

As $\omega_d \to \omega_d^{\text{asym}+}$ (approaching the asymptote from the right), $a\omega_d + b \to 0^-$.
Hence

$$\frac{A}{a\omega_d + b} \to -\infty \tag{28}$$

and therefore

$$f(\omega_d) \to -\infty \tag{29}$$

These limiting behaviors imply two important results. First, there exists an interval immediately to the **left** of the asymptote where the objective function is strictly positive, since the fractional component becomes arbitrarily large. Second, the function becomes strictly negative immediately to the **right** of the asymptote. Thus, the objective function has a positive region for $\omega_d < \omega_d^{\text{asym}}$ and a negative region for $\omega_d > \omega_d^{\text{asym}}$.

2.7.3. Only One Critical Point Produces a Positive Objective Value

From the first-order optimality condition derived in Sub-section 1, the stationary points satisfy

$$a\omega_d + b = \pm r, r = \sqrt{\frac{A(-a)}{s}} > 0 \tag{30}$$

Solving for ω_d yields the two candidate critical points reported in equations (22) and (23):

$$\omega_{d,1} = \frac{\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)}{-\left(k_e - k_d(1-T)\right)} \tag{22}$$

$$\omega_{d,2} = \frac{-\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)}{-\left(k_e - k_d(1-T)\right)} \tag{23}$$

To determine which of these stationary points yields a positive objective value, rewrite the function in terms of

$$y = a\omega_d + b \tag{31}$$

giving

$$f(y) = \frac{A}{y} - \frac{s}{a}y + \left(C + \frac{s}{a}b\right) \tag{32}$$

Define

$$k := \frac{s}{-a} > 0, \qquad r = \sqrt{\frac{A}{k}}$$

Evaluating the function at the two stationary points:
At $y = +r$ (corresponding to $\omega_{d,1}$):

$$f(r) = C - kb \tag{33}$$

At $y = -r$ (corresponding to $\omega_{d,2}$):

$$f(-r) = C - kb - 2\sqrt{Ak} \tag{34}$$

Since $2\sqrt{Ak} > 0$, we have

$$f(\omega_{d,2}) = f(\omega_{d,1}) - 2\sqrt{Ak} < f(\omega_{d,1}) \tag{35}$$

Moreover, from Sub-section 2 we know that the region immediately to the left of the asymptote produces positive values of the objective function. The only critical point located on that side is $\omega_{d,1}$. Hence:
$\omega_{d,1}$ lies in the region where positive objective values are possible.
$\omega_{d,2}$ lies in the negative region and thus cannot produce a positive objective value.
Therefore, the only stationary point that can produce a positive objective value is:

$$\omega_d^* = \frac{\sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}} - (k_e - g)}{-(k_e - k_d(1-T))}$$

$$or$$

$$\omega_d^* = \frac{(k_e - g) - \sqrt{\frac{F(1+g)[k_e - k_d(1-T)]}{D+E}}}{(k_e - k_d(1-T))} \tag{36}$$

which corresponds to equation (22).

2.7.4. Second-Order Condition: The Positive-Valued Stationary Point is a Local Minimum

The second derivative of the objective function after substitution is

$$f''(\omega_d) = \frac{2Aa^2}{(a\omega_d + b)^3} \tag{37}$$

Recall that:
$a = k_d(1-T) - k_e < 0$ implies $a^2 > 0$,
the sign of $f''(\omega_d)$ depends entirely on the sign of $(a\omega_d + b)^3$,
at $\omega_{d,1}$,
$a\omega_{d,1} + b = +r > 0$
at $\omega_{d,2}$,
$a\omega_{d,2} + b = -r < 0$
Thus:
At the *positive-valued critical point* $\omega_{d,1}$:
$f''(\omega_{d,1}) = \frac{2Aa^2}{r^3} > 0$
indicating a local minimum.
At the critical point $\omega_{d,2}$:
$f''(\omega_{d,2}) = \frac{2Aa^2}{(-r)^3} < 0$
indicating a local maximum on the right side of the asymptote.
Combining this with the previous findings:

The only critical point capable of producing a positive objective value, $\omega_{d,1}$, is a **local minimum**, not a maximum.
The true maximum in the positive region is not attained at a stationary point but instead occurs as $\omega_d \to \omega_d^{\text{asym}-}$, where

$$f(\omega_d) \to +\infty \tag{38}$$

Hence, although one stationary point yields a positive objective value, it corresponds to a local minimum. The other stationary point is a local maximum but lies entirely in the negative region of the domain.

2.8. Model Validation Using Simulated Numerical Inputs

This chapter validates the theoretical model developed in Chapters 3 and 4 by applying simulated or randomly generated numerical values. The purpose of this numerical experiment is to demonstrate the internal consistency of the model, verify the analytical behavior of the objective function, and confirm that the critical-point expression derived in equation (22) behaves as expected under controlled conditions.

3. Result and Discussion

3.1. Result

3.1.1. Parameter Setup for Numerical Simulation

To conduct model testing under simulation, this study assigns plausible numerical values to all relevant parameters of the model. These values do not refer to any real-world firm but are constructed to satisfy fundamental economic conditions, namely:
Cost of debt < cost of equity
Non-negative growth rate
Positive discount factors
$D + E \geq C$(operational cost condition)
$k_e > k_d(1 - T)$ to ensure $a = k_d(1 - T) - k_e < 0$
A typical set of simulated inputs may include:

Table 2. Numerical input

Parameter	Symbol	Value	Description
Cost of debt	k_d	0.07	Annual before-tax debt cost
Cost of equity	k_e	0.12	Expected return on equity
Corporate tax	T	0.22	Effective tax rate
Growth rate	g	0.02	Long-term growth
Free cash flow	F	1	Scaling constant
Cash	C	3	Cash and cash equivalents
Debt	D	5	Total Debt has interest rate
Equity	E	5	Total Equity

Table 3. Debt optimization output from simulated numerical inputs.

Output	ω_d	Equity Value
ω_d current	0.50	13.16
ω_d max	1.00	22.48
ω_d min	0.00	13.20
ω_d critical point selected	0.28	12.69
ω_d selected	**1**	**22.48**

Based on the numerical inputs in Table 2 and the simulation outputs in Table 3, the model exhibits behavior fully consistent with the theoretical findings. The equity value at the current leverage level ($\omega_d = 0.50$) is 13.16, while the equity value at zero leverage ($\omega_d = 0.00$) is slightly higher at 13.20.

When the firm increases its leverage to the upper bound ($\omega_d = 1$), the equity value rises substantially to 22.48.

The critical point derived from the first-order condition occurs at $\omega_d = 0.28$, producing an equity value of 12.69, which is lower than the values at surrounding leverage levels. This confirms the theoretical prediction that the critical point represents a local minimum, not a maximum. The highest equity value in the simulation appears at $\omega_d = 1$, in line with the analytical result that the objective function grows as the debt ratio approaches the left-hand side of the asymptote.

Overall, the simulated results support the conclusion that the model has no interior maximum, and that the maximum equity value within the simulated domain occurs at the highest feasible leverage. This behavior is visualized in Figure 1, which illustrates the shape and direction of the objective function. On Figure 1, point A is ω_d critical point selected, point B is ω_d critical points that result in negative EV values, and point C is ω_d that result maximum equity value. The y-axis is equity value and x-axix is ω_d.

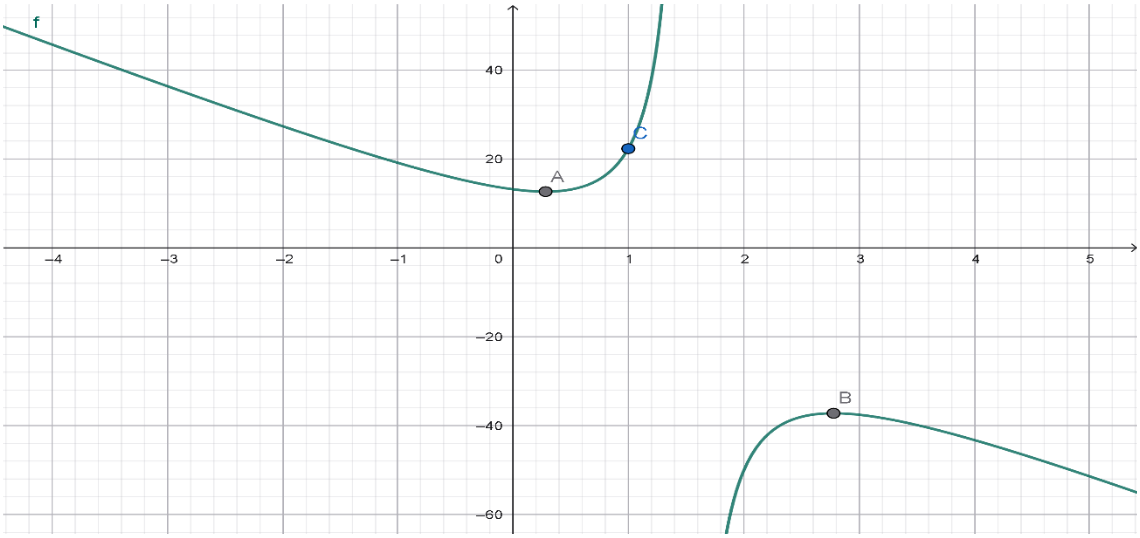

Figure 1. Result Graph

3.1.2. Results and Model Validation on Real Case

1) Data and Firm-Level Characteristics
   To illustrate the empirical foundations of our modeling procedure, we construct a focused dataset drawn from two countries and two distinct industries. We begin by selecting firms that provide five consecutive years of complete financial statements, ensuring that the variables required by the model such as EBITDA, free cash flow, leverage, and cost of capital are consistently observable. We then identify five representative firms from the technology sector and five from the consumer sector in both the United States and Indonesia. While this set of twenty firms is intentionally restricted in size, it provides sufficient time-series structure at the firm level to evaluate the dynamic behavior of our optimization framework. As in other structurally oriented empirical studies, one would expect that expanding the analysis to a broader population of firms would introduce additional cross-sectional variation, thereby strengthening rather than weakening the general applicability of the results.

To be consistent with the structural logic applied in benchmark capital structure models such as the framework developed by Goldstein, Ju, and Leland (2001) [7], we evaluate firms using measures tied directly to operating cash flow stability and long-run value dynamics.

The following criteria governed inclusion in the sample:

1) EBITDA Stability Rule

A company was included only if its CV of EBITDA fell below the sector-country average:

a. United States
   - Consumer : CV ≤ 0.08

- Technology : CV ≤ 0.42

b. Indonesia
- Consumer : CV ≤ 0.20
- Technology : CV ≤ 0.39

Cash-Generation Classification (Percentile Method)
Using a several firm industry reference set:
- High cash-generating: Above 75th percentile
- Low cash-generating: Below 25th percentile

Terminal Growth Rate (TGR) Rule
TGR constrained within macroeconomic limits:
- Indonesia : 1.5%–2.5%
- United States: 2%–4%

Rules :
- If EBITDA CAGR exceeded range → TGR = midpoint
- If within range → TGR = CAGR
- If below range → TGR = lower bound

After ensuring compliance with these selection rules, the compiled firm-level dataset can then be presented as follows.

Table 4. Firm selected

Country	Sector	Firm	FCFF/Assets	CV EBITDA	TGR Selected
US	Technology	Amazon	Low	0,40	3,00%
US	Technology	Microsoft	Medium	0,26	3,00%
US	Technology	Tesla	Medium	0,43	3,00%
US	Technology	Meta	Medium	0,31	3,00%
US	Technology	Apple	High	0,07	3,00%
US	Consumer	PepsiCo	Medium	0,08	3,00%
US	Consumer	Walmart	Low	0,08	3,00%
US	Consumer	P&G	Medium	0,07	3,61%
US	Consumer	Nestlé	Low	0,04	2,00%
US	Consumer	Kraft Heinz	High	0,05	2,00%
ID	Technology	MTDL	Low	0,24	2,00%
ID	Technology	MLPT	Medium	0,14	2,00%
ID	Technology	TLKM	Medium	0,05	2,50%
ID	Technology	ISAT	Low	0,31	2,00%
ID	Technology	DMMX	High	0,35	2,00%
ID	Consumer	ICBP	High	0,20	2,00%
ID	Consumer	HMSP	Low	0,12	2,50%
ID	Consumer	ULTJ	Medium	0,10	2,50%
ID	Consumer	AALI	Medium	0,16	2,50%
ID	Consumer	INDF	High	0,18	2,00%

2) Empirical Results (EV-Based Model Output)

a. The implementation of the Lagrange optimization model for the twenty firms in our dataset requires several key inputs derived from the firms' five-year financial histories. These inputs include EBITDA, free-cash-flow-to-asset ratios, volatility measures (CV EBITDA), cost of equity, cost of debt, and the firm-specific terminal growth rate assigned using the convergence rule described previously. Using these parameters, the model computes the firm's optimal debt weight (wd*) and then recalculates enterprise value (EV) under this recommended capital structure.

b. To evaluate whether the model's recommendation is economically sound, we compare the Current Enterprise Value which reflects the firm's existing capital structure to the EV Potential, which represents the enterprise value the firm would have if it adopted the model's recommended wd*. The criterion for validation is straightforward and intuitive:

c. If EV Potential > Current EV, then the model's recommended capital structure is value-increasing.
d. This comparison provides a direct economic test of the model. Instead of relying solely on theoretical coherence or mathematical elegance, the model must show that its recommendation improves firm value in practice. The difference between these two values, expressed as a percentage of Current EV, is referred to as the EV Upside and is used as the primary measure of empirical validation.

Table 5. EV Comparison Under Current and Model-Implied Capital Structures.

Country	Sector	Firm	WACC	Current EV (Mill USD)	EV Potential (Mill USD)	Upside (%)
US	Technology	Amazon	9,63%	568.063	613.190	7.9%
US	Technology	Microsoft	8,88%	1.297.508	1.502.449	15.8%
US	Technology	Tesla	8,65%	96.522	103.202	6.9%
US	Technology	Meta	7,82%	881.047	918.827	4.3%
US	Technology	Apple	6,57%	3.197.206	3.197.206	0.0%
US	Consumer	PepsiCo	5,68%	361.978	361.978	0.0%
US	Consumer	Walmart	6,78%	246.554	304.324	23.4%
US	Consumer	P&G	6,54%	563.503	705.104	25.1%
US	Consumer	Nestlé	6,71%	144.275	176.012	22.0%
US	Consumer	KraftHein	4,69%	100.713	120.006	19.2%
ID	Technology	MTDL	8,55%	767	822	7.2%
ID	Technology	MLPT	7,90%	291	374	28.7%
ID	Technology	TLKM	7,01%	46.680	57.792	23.8%
ID	Technology	ISAT	9,73%	6.483	7.290	12.4%
ID	Technology	DMMX	8,03%	204	287	40.7%
ID	Consumer	ICBP	7,86%	11.534	12.056	4.5%
ID	Consumer	HMSP	9,05%	6.024	9.273	53.9%
ID	Consumer	ULTJ	8,11%	1.080	1.396	29.3%
ID	Consumer	AALI	8,11%	1.256	1.259	0.2%
ID	Consumer	INDF	10,9%	9.196	11.277	22.6%

The EV comparison shows that 18 out of 20 firms experience positive EV gains when adopting the model's recommended capital structure. This provides strong empirical evidence that the model captures value-enhancing adjustments for the majority of companies across different sectors and countries. The magnitude of the EV gains for example, 23–25% for large U.S. consumer firms or more than 40% for firms like DMMX in Indonesia indicates that the model does not merely produce statistical improvements but economically meaningful changes in enterprise value.

Two firms, however Apple and PepsiCo exhibit 0.0% EV Upside, meaning that the model's recommended wd* does not produce any additional value beyond the firm's current capital structure. Importantly, this does not count against the model; in fact, it reinforces its validity. These cases indicate that these firms are already operating near their optimal capital structure, given their stable cash flows, low marginal tax-shield benefits, and already-efficient financing mix. A valid model should not force a change when no economic benefit exists, and in these two firms, the model correctly identifies that any further leverage adjustment would not meaningfully alter enterprise value.

Thus, the absence of gains in these firms is itself a sign of correctness. A flawed model would insist on excessive leverage changes for all firms regardless of context, while a valid model recognizes when the current financing policy is already optimal or close to optimal. Apple and

PepsiCo serve as benchmark examples showing that the model is sensitive to firm-specific conditions rather than mechanically producing uniform recommendations.

Taken together, the results from this section confirm that the Lagrange optimization framework produces capital-structure recommendations that are both theoretically consistent and empirically value-enhancing, thereby supporting the model's validity.

3.1.3. Theoretical Validity

The model is built on the premise that firm value is jointly shaped by operating cash-flow characteristics, financing frictions, and the sensitivity of WACC to changes in the debt ratio. Through these interacting channels, the optimal debt weight arises endogenously, offering a coherent foundation for understanding how capital-structure decisions evolve within the optimization setting.

A key source of internal consistency appears in the Trade-Off Theory, which highlights the balance between debt tax shields and expected distress costs. Under stable operating conditions, the model naturally increases optimal leverage to lower WACC, while higher volatility induces a shift toward more conservative debt ratios. This behavior mirrors classic trade-off logic without requiring manual calibration of distress parameters.

The framework also incorporates insights from Free-Cash-Flow Theory and Pecking Order Theory. While cash-flow efficiency shapes the incentive to employ leverage, the eventual valuation impact depends on structural elements such as initial leverage gaps, relative financing costs, and regulatory borrowing limits. Likewise, the model's cautious recommendations for high-uncertainty firms reflect the informational frictions emphasized in Pecking Order Theory.

Within this structure, the FCF-to-Asset classification captures differences in cash-flow efficiency but does not predetermine the magnitude of value creation. Empirically, some low-efficiency firms such as HMSP show substantial EV gains (53,9%), while several high-efficiency firms experience only modest improvements. This pattern is consistent with capital-structure theory, which holds that firms farther from their optimal leverage tend to exhibit larger valuation adjustments [8] [9], and that higher free cash flow strengthens the incentive to employ leverage without guaranteeing proportional value gains [10].

Overall, these patterns suggest that the model operates in a way that is generally consistent with foundational capital-structure principles, providing a reasonable basis for examining how leverage adjustments influence firm value across differing operating environments.

4. Conclusion

This study develops a capital structure optimization model aimed at determining the optimal debt ratio through a mathematical approach based on the Lagrange method. By formulating equity value as a function of the proportion of debt, the analysis demonstrates that the behavior of the objective function is fundamentally shaped by the presence of a vertical asymptote separating the leverage domain into regions of positive and negative equity value. This structural characteristic implies that the objective function lacks any interior maximum, contradicting the traditional assumption that capital structure exhibits a well-defined optimum within the feasible range.

The mathematical results show that two critical points arise from the first-order condition. However, only one of these points lies within the positive region of the objective function, and the second-order derivative confirms that this point is a local minimum, not a maximum. The second critical point, located in the negative region, corresponds to a local maximum that has no economic relevance. These findings mathematically refute the central claim of the Trade-Off Theory, which asserts the existence of an interior optimal capital structure where tax benefits offset expected financial distress costs.

Numerical simulations and real-world validation reinforce the analytical structure of the model. Equity value increases sharply as leverage approaches the asymptote from the left, while minimum leverage (zero debt) still produces stable and positive equity value. These outcomes indicate that optimal equity value can be achieved under two extreme conditions:

1) Minimum leverage (no debt) for conservative financial strategies, and

2) High leverage approaching the asymptote for aggressive value-maximizing strategies.

This dual-optimality gives firms flexibility to align leverage policies with their risk profiles, market environments, and managerial preferences.

Overall, the study concludes that:

1) The Lagrange-based optimization model successfully provides an explicit mathematical formulation linking leverage and equity value.
2) The objective function is non-standard, attaining its supremum near the asymptote rather than at an interior point.
3) The interior critical point is not optimal, refuting the traditional Trade-Off Theory.
4) Optimal equity value arises at two leverage extremes: zero leverage or high leverage near the asymptote.
5) Simulated and empirical results confirm the internal consistency and stability of the model.
6) This study lays the groundwork for a new theoretical and econometric framework for capital structure analysis, particularly in evaluating extreme leverage and identifying non-feasible leverage regions.
7) The model also provides practical guidance for firms in setting capital structure priorities, supporting both conservative and aggressive financing strategies.

Despite these contributions, the model remains limited by its deterministic structure, its exclusion of bankruptcy risk, agency costs, and regulatory constraints, and by a narrow empirical sample. Future research may incorporate stochastic risk components, dynamic parameters, capital-market frictions, and broader cross-industry datasets. These enhancements would not only strengthen empirical robustness but also transform this model into a more advanced econometric tool. Importantly, the model presented in this study serves as a foundational framework upon which more complex econometric models of capital structure can be built and systematically expanded.

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